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LETTER TO THE EDITOR

A particle system with massive destruction

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Abstract. We consider an interacting particle system on the one-dimensional integer lattice in which vacant sites become occupied at rate λ, and clusters of adjacent occupied sites die at a rate equal to their size. We establish the existence and uniqueness of an invariant measure when λ is less than 1, in addition to establishing that the distribution of cluster sizes in equilibrium decays exponentially.

In this letter we consider an interacting particle system on the one-dimensional integer lattice Z in which vacant sites become occupied at rate λ and clusters of n occupied adjacent sites become simultaneously vacant (or 'die') at rate n. We will refer to this system as the cluster model. This can be pictured as follows: trees are born at vacant sites at rate λ, and lightning strikes sites at rate 1. When lightning strikes any tree in a cluster, the entire cluster instantly catches fire and burns down. We show that for λ between 0 and 1, there exists a unique invariant measure. A corollary to the proof of this result is that the distribution of cluster sizes in the invariant measure decays exponentially.

Systems such as the cluster model discussed here are suggested by certain cellular automata, which are reminiscent of the evolution of sandpiles ([1, 2]). Numerical studies reveal that a wide range of automata evolve to an attractor which displays large fluctuations in avalanche size, indicative of behaviour of systems near criticality. The models all have the property that any site can be occupied by an arbitrary number of particles (grains of sand)—an attribute which so far has impeded rigorous analysis of all but the simplest models (which do not exhibit this interesting behaviour).

Consequently, it is our intention to examine models which can exhibit large events on the relatively simple state space $X = \{0, 1\}^Z$. We will denote the configuration of the cluster process at time t by η_t, where $\eta_t(x) = 1$ means that the site x is occupied at time t, where $\eta_t(x) = 0$ means the site is vacant. Recently, we learned that numerical simulations had been done on similar models by the same groups mentioned above [3], as well as by Henley [4].

To state our result concretely, we first restrict the state space to the subset of X consisting of configurations with an infinite number of zeros on either side of the

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origin. We denote this subset of configurations by X':

$$X' = \left\{ \eta \in X: \sum_{x \leq 0} (1-\eta(x)) = \sum_{x \geq 0} (1-\eta(x)) = \infty \right\}.$$

(It is clear that an infinite cluster would die instantly, and this behaviour would yield continuity problems which are not germane to the questions we would like to address.) In this state space, we define our process by prescribing the following rates:

(i) vacant sites become occupied at rate 1;

(ii) clusters of n adjacent occupied sites become vacant simultaneously at rate n.

It is clear that this process is not attractive or reversible. Furthermore, there does not appear to be a natural dual process ([5]). The central result that we prove in this letter is encapsulated in the following theorem.

Theorem 1. When $\lambda < \frac{1}{2}(\sqrt{5}-1)$ there exists a unique invariant measure on X' for the cluster process η_t.

Remark. This upper limit on λ in theorem 1 is a consequence of the ease of exposition. At the end of this letter we describe how to extend the upper bound to a number greater than 1.

Denote the invariant measure announced in theorem 1 by μ. Then a consequence of the proof of theorem 1 is the following corollary.

Corollary of the proof. Denoting the size of the cluster surrounding x by $N(x)$, we have

$$\mu(N(x)=n) \leq n\left(\frac{\lambda}{\lambda+1}\right)^n.$$

Proof. We begin by making a brief remark on the construction of the process η_t (details will follow in a subsequent paper). Denote the set of bounded continuous functions on X' by $C(X')$. A sequence of functions $f_n \in C(X')$ converges to $f \in C(X')$ if $\sup_n \|f_n\| < \infty$ and if $f_n \to f$ uniformly on compact sets, where $\| \ \|$ denotes the uniform norm. We construct a semigroup on $C(X')$ as a limit of process on finite intervals.

Let $Z_{m,n} = \{m, m+1, \ldots, n\}$, and let $X_{m,n} = \{0,1\}^{Z_{m,n}}$. Denote the set of all functions on $X_{m,n}$ by $C_{m,n}$. We define a process $\eta_t^{m,n}$ with state space $X_{m,n}$, which evolves just like η_t described above, with the assumption that $\eta_t^{m,n}(m-1) = \eta_t^{m,n}(n+1) = 0$. It can be shown that a well defined limiting process exists by taking $m \to -\infty$ and $n \to \infty$ (the construction is essentially identical to that which appears in chapter VII, section 3 of [5]).

The centrepiece of the proof of theorem 1 is the following graphical representation of the process (see figure 1). With each site $x \in Z$ are associated two Poisson processes $B_n(x)$ at rate λ, and $D_n(x)$ at rate 1. The arrival times of the $B_n(x)$ process (full circles (●) in figure 1) will denote birth times at site x, so that if site x is vacant and a birth time occurs, site x becomes occupied. The arrival times of the $D_n(x)$ process (crosses (×) in figure 1) denote death marks, which have the property that when site x is occupied and a death mark occurs, site x and all sites in the cluster containing x become vacant. One should picture a horizontal integer lattice representing the spatial location of each site, and emanating from each site is a vertical coordinate representing time.

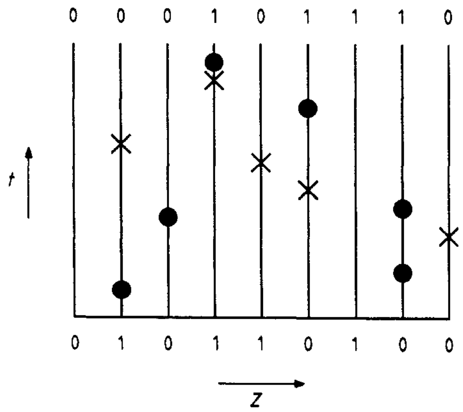

Figure 1. A realisation of the graphical representation is shown, along with its action on the initial configuration shown at the bottom of the figure.

We will use a coupling argument based on the graphical representation to show the existence of a unique invariant measure. The basic coupling consists of first placing the birth and death marks according to the B_n and D_n processes on the time axes, and then running the two processes η_t and ζ_t using these marks. In this way the marginal distributions of (η_t, ζ_t) are the ones associated with the uncoupled process, but the evolutions of the two coupled processes are related via the graphical representation.

We will also construct an independent process ξ_t using the graphical representation. In this process, births occur at rate λ and deaths occur at rate 1 independently at each site. The same graphical representation used for the cluster model can be used for ξ_t by declaring that the effect of a death mark is to vacate only the site on which the mark appears (as opposed to removing the entire cluster associated with that site). With this coupling it is clear that if η_t and ξ_t are started from the same initial configuration, or, more generally, if $\eta_0 \leqslant \xi_0$ (i.e. $\eta_0(x) \leqslant \xi_0(x)\ \forall x$), then $\eta_t \leqslant \xi_t$ for all t with the usual ordering.

The existence of an invariant measure μ from the usual soft arguments involving compactness fail here, due to the fact that the state space X' is not compact. However, the proof of existence is actually an immediate consequence of coupling between the cluster process η_t and the process ξ_t in which births and deaths occur at each site independently and at rate 1.

As stated above, if $\eta_0 \leqslant \xi_0$, then $\eta_t \leqslant \xi_t$ for all t. The process η_t is a Feller process on X', and the process started from any distribution of configurations in X' remains supported on X' almost surely for all subsequent times, due to the fact that the same is trivially true for the ξ_t process. Since $X' \subset X = \{0, 1\}^Z$ which is compact, there exists at least one invariant measure on X'.

We will prove ergodicity by establishing that for any two initial configurations η_0 and ζ_0, given any interval of sites $Z_{m,n}$, then:

$$\lim_{t\to\infty} P(\eta_t(x) = \zeta_t(x)\ \forall x \in Z_{m,n}) = 1. \tag{1}$$

This will imply ergodicity (see [5] p 130), since, by taking any bounded continuous function f on X', and letting μ be an invariant measure, we obtain

$$\int f\,\mathrm{d}\mu - E^{\zeta}f(\zeta_t) = \int [E^{\eta}f(\eta_t) - E^{\zeta}f(\zeta_t)]\mu(\mathrm{d}\eta)$$

which tends to zero by (1) as $t \to \infty$, where E^ζ denotes expectation with respect to the measure on processes starting with initial configuration ζ. Therefore,

$$\lim_{t \to \infty} E^\zeta f(\zeta_t) = \int f \, \mathrm{d}\mu.$$

To prove (1) we consider η_t and ζ_t coupled via the graphical representation. The idea is to define left and right boundaries of a region containing the origin on which the two processes agree; no claim is made as to agreement or disagreement outside this region. From now on we will concentrate on the right boundary, the left boundary will evolve in an analogous manner. The picture to have in mind is the two configurations written in the following way:

$$\eta_t \ldots 00110011|1010\ldots$$

$$\zeta_t \ldots 00110011|0100\ldots$$

where the symbol | occurs just after the location of the right edge, denoted R_t. In this example, if the next event is the appearance of a death mark at the site R_t, then the edge can move right by one unit. If, however, the next event is a death mark at the site $R_t + 1$, then the edge must move left two sites to find agreement between the two configurations (in fact, our rules may have the edge move farther to the left in this case). Note that 'disagreement' cannot propagate beyond the first common zero.

The precise rules for the evolution of R_t will make R_t a random walk with positive drift. For reasons that will become clear, we will require that the right edge move right by leaving zeros in its wake. Both processes will be simultaneously equal to zero at the origin at an infinite sequence of times (for example, whenever a death mark hits the origin). Therefore, any time either of the edges returns to the origin, we can wait until the next death mark hits the origin and restart the process. With probability one, eventually the transience of the left and right edges will yield agreement on any interval for all sufficiently large times.

We begin by defining the following stopping times which indicate the occurrence that $\eta_t(0) = 0$ and $\zeta_t(0) = 0$ after a disagreement:

$$\tau_1 = \inf\{t > 0 \colon \eta_t(0) = \zeta_t(0) = 0\}$$

$$\tau_i = \inf\{t > \tau_{i-1} \colon \eta_t(0) = \zeta_t(0) = 0 \text{ and } \exists s \in (\tau_{i-1}, t) \colon \eta_s(0) \neq \zeta_s(0)\}.$$

At time τ_1 we set R_t to be the location of the rightmost site so that all points between the origin and R_t are zeros

$$R_{\tau_1} = \sup\{x \geq 0 \colon \eta_{\tau_1}(z) = \zeta_{\tau_1}(z) = 0 \forall z \colon 0 \leq z \leq x\}.$$

As we have said, the left edge is to be dealt with in a symmetric fashion. We now prescribe the evolution of R_t.

Denote the time of the ith jump of R_t by σ_i, where we take $\sigma_1 = \tau_1$, and the position of R_t after at time σ_i by R_{σ_i}. We begin by specifying a set of events $A^n_{x,t}(j)$, indexed by j, which, for any values of n, x and t, are disjoint. These events will involve events at sites $y \colon x \leq y \leq x + n$ occuring at times after t for some finite n. These events are constructed so that the occurrence of any of the $A^n_{x,t}(j)$ moves R_t right by at least one step, in such a way that the new sites of agreement are all occupied by zeros. If none of these events occur, then by default the edge will move left (the distance that it jumps left will be discussed momentarily).

The idea is to fix a value of n, and find a range of λ so that the drift of R_{σ_i} is positive. The simplest case is $n=1$, where only the site one unit to the right of R_{σ_i} is considered. In this case there is only one good event:

$$A_i^1 = \{\text{death at } R_{\sigma_i}+1 \text{ before birth at } R_{\sigma_i}\}$$

where A_i^1 denotes $A_{x,t}^1$ with $x=R_{\sigma_i}$ and $t=\sigma_i$. If this event occurs, agreeing zeros are created one unit to the right of the original position of R_{σ_i}, and, consequently, R_t moves right. No disagreements could have been created to the left of R_{σ_i}, since the zeros at $\eta_t(R_{\sigma_i})$ and $\zeta_t(R_{\sigma_i})$ 'insulate' the sites to the left of R_t from any events to the right. If the complement of A_i^1 occurs, i.e. a birth at R_{σ_i}, then we can no longer guarantee that events to the right of R_{σ_i} will not affect agreement, and we are forced to move R_t to the left. notice that, in any case, death marks to the left of R_t can never cause disagreement to the left of R_t.

In general, with a fixed n, we will denote good events (those which move R_{σ_i} right) by:

$$G_{i,n} = \cup_j A_i^n(j)$$

and bad events (those which move R_{σ_i} left) by $G_{i,n}^c$. Then, σ_{i+1} is the time of the first occurrence of $G_{i,n}$ or $G_{i,n}^c$.

Recall that the independent process ξ_t, when started from the same configuration as the cluster process η_t, dominates η_t ($\xi_t(x) \geq \eta_t(x)$ for all x). This motivates the following scheme for moving R_t to the left. At the instant the right edge passes a point x, we start a process $\xi_t(x)$ in which the birth and death marks at x act only on x. Then, if none of the (good) events comprising $G_{i,n}$ occur, we will move the right edge left to the first site x on which the process $\xi_t(x)$ is zero. Since each of these processes is independent, and each started with initial value zero (recall the right edge moves right by leaving a trail of agreeing zeros), we will be able to bound the probability that the jump to the left exceeds $n+1$ sites by $[\lambda/(\lambda+1)]^n$.

To make this precise, we begin by identifying the time when the right edge R_t last passed from left to right through site x:

$$f_{t,x} = \sup\{s\colon s<t \text{ such that } R_s<x\}.$$

Note that $f_{t,x}=\tau_1$ if $R_s>x$ for all $s\in[\tau_1, t]$.

We now define the processes $\xi_t^i(x)$ using the graphical representation, for all (x, t) such that

$$\text{(i) } x<R_{\sigma_i} \qquad \text{(ii) } t>f_{t,x}$$

so that a death mark at x induces the transition $1\to 0$, and a birth mark induces $0\to 1$. Since $\eta_{f_{t,x}}(x)=0$ for all $x<R_{\sigma_i}$, we will take $\xi_{f_{t,x}}^i(x)$ distributed as a product measure with $P(\xi_{f_{t,x}}(x)=1)=\lambda/(\lambda+1)$ (so that $\eta_{f_{t,x}}(x)\leq \xi_{f_{t,x}}^i(x)$, $\forall x\colon 0\leq x\leq R_t$). Then the configurations of $\xi_t^i(x)$ and $\xi_t^i(y)$ are independent for $x<y<R_{\sigma_i}$ and for $t\colon \sigma_i\leq t<\sigma_{i+1}$. (It would be simpler to just take $\xi_{f_{t,x}}^i=0$, but then the sites of ξ^i are not uncorrelated, although the correlations only help us.)

We will now consider the good event $G_{i,1}$ which moves R_{σ_i} right by considering events only at sites R_{σ_i} and $R_{\sigma_i}+1$. We can write the following formula for the jump that the R_t makes at σ_{i+1}, which we denote by $Z_{\sigma_{i+1}}$:

$$Z_{\sigma_{i+1}} = \inf\{x-R_{\sigma_i}\colon \xi_{\sigma_{i+1}^-}^i(z)=1\ \forall z\colon x<z\leq R_{\sigma_i}\}1_{\{G_{i,1}^c\}} + 1_{\{G_{i,1}\}} \tag{2}$$

where we have used the notation t^- to denote the limit of an increasing sequence of times approaching t. The indicator functions in the above definition serve to distinguish between the events where R_t moves left or right, and the remaining factor determines the size of the associated jump left. We can now write the position of R_t as $R_{\sigma_{i+1}} = R_{\sigma_i} + Z_{\sigma_{i+1}}$ if this quantity is non-negative; otherwise we restart the argument at the next τ_i.

Recalling that $P(\xi^i_{\sigma_i^-+1}(x)=1) \le \lambda/(\lambda+1)$, that these events are independent of $G_{i,1}$, and that $P(G_{i,1}) = 1/(\lambda+1)$, we know that R_{σ_i} is a discrete-time random walk with drift

$$E(Z_{\sigma_{i+1}}) \ge \left(\frac{1}{\lambda+1}\right) - (\lambda+1)\left(\frac{\lambda}{\lambda+1}\right)$$

where the first term on the right-hand side comes from bounding the jump size to the right by one, and the second term is obtained by summing the geometric series for the expected jump size left. Requiring that $E(Z_{\sigma_{i+1}}) > 0$, we find that if $\lambda \in [0, (\sqrt{5}-1)/2)$, then R_t is transient, completing the proof.

An obvious way to improve this bound is to consider events involving more sites. (In this case equation (2) is generalised to allow for forward jumps of more than one site.) For example, suppose we look at sites R_{σ_i}, $R_{\sigma_i}+1$ and $R_{\sigma_i}+2$, and see exactly the following events: a death at $R_{\sigma_i}+2$ followed by a birth at R_{σ_i}, and then a death at R_{σ_i}. Since the death at $R_{\sigma_i}+2$ insulates the right edge from death marks further to the right, we can move R_{σ_i} forward, whereas if we had only been looking at R_{σ_i} and $R_{\sigma_i}+1$, we would have been forced to move the right edge left.

Considering (a subset of the possible) events on the five sites $R_{\sigma_i}, \ldots, R_{\sigma_i}+4$, we are able to show that the process is ergodic for $\lambda \le 1.01$. The proof is exactly analogous to the above proof, except that it is much more messy. While further improvements are possible, the returns on the labour involved are rapidly diminishing.

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